J. Group Theory **17** (2014), 661–666
DOI 10.1515/jgt-2013-0056

Counting conjugacy classes of cyclic subgroups for fusion systems

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Communicated by Radha Kessar

Abstract. Thévenaz [6] made an interesting observation that the number of conjugacy classes of cyclic subgroups in a finite group G is equal to the rank of the matrix of the numbers of double cosets in G. We give another proof of this fact and present a fusion system version of it. In particular we use finite groups realizing the fusion system $\mathcal{F}$ as in our previous work [3].

1 Statements of the results

In [6], Thévenaz observed the 'curiosity' that a finite cyclic group G can be characterized by the nonsingularity of the matrix of the numbers of double cosets in G. In fact he proved a stronger proposition that for an arbitrary finite group G the number of the conjugacy classes of cyclic subgroups of G is equal to the rank of that matrix. This can be stated slightly more generally by introducing a subgroup H of G and considering the G-conjugacy classes of subgroups of H as follows.

Theorem 1.1. *Let G be a finite group and let $H \le G$. The rank of the matrix $(|P\backslash G/Q|)_{P,Q\le_G H}$, whose rows and columns are indexed by the G-conjugacy classes of subgroups of H and whose entries are the numbers of the corresponding double cosets in G, is equal to the number of the G-conjugacy classes of cyclic subgroups of H.*

In the above theorem, the matrix $(|P\backslash G/Q|)_{P,Q\le_G H}$ is determined by the (H,H)-biset ${}_HG_H$, i.e., the set G with left and right H-action (induced by multiplication in the group G) which are compatible. In particular, when $H = S$ is a Sylow p-subgroup of G, the (S,S)-biset ${}_SG_S$, viewed as a left $S\times S$-set, decomposes as

$$ {}_SG_S \cong \coprod_{x\in[S\backslash G/S]} S\times S/\Delta(c_x, S\cap S^x). $$

See Section 3 for the notation. Consequently the (S,S)-biset ${}_SG_S$ determines the p-fusion of G, i.e., the conjugacy relation between p-subgroups of G. The essential feature of the p-fusion in finite groups is generalized to categories called *saturated fusion systems*, which connect the p-local aspects of group theory, representation theory and homotopy theory. We refer the reader to the book [1] for an introduction to the subject.

In [3], we observed that every saturated fusion system $\mathcal{F}$ on a finite p-group S can be realized by a finite group G containing S as a (not necessarily Sylow) p-subgroup. Thus the above theorem yields a fusion system version as follows.

Theorem 1.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let G be a finite group which contains S as a subgroup and realizes $\mathcal{F}$. Then the rank of the matrix $(|P\backslash G/Q|)_{P,Q\leq_G S}$ is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.*

By a result of Broto, Levi and Oliver [2, Proposition 5.5], every saturated fusion system $\mathcal{F}$ on a finite p-group S has a (non-unique) characteristic biset Ω. See Section 3 for a precise definition; in particular, Ω is a finite (S,S)-biset. If $\mathcal{F}$ is the fusion system of a finite group G on its Sylow p-subgroup S, then G is a characteristic biset for $\mathcal{F}$ with the obvious S-action on the left and right. So we may well expect that the matrix of the above theorem, with G replaced by Ω, has the same rank. Indeed this is the case.

Theorem 1.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let Ω be a characteristic biset for $\mathcal{F}$. Then the rank of the matrix $(|P\backslash\Omega/Q|)_{P,Q\leq_{\mathcal{F}} S}$ of the number of (P,Q)-orbits of Ω indexed by the $\mathcal{F}$-conjugacy classes of subgroups of S is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.*

Finally, one can replace the characteristic biset Ω in the above theorem by the characteristic idempotent $\omega_{\mathcal{F}}$ (which is a virtual (S,S)-biset; see Section 3) with $|P\backslash\omega_{\mathcal{F}}/Q|$ as the linearized number of (P,Q)-orbits.

Theorem 1.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $\omega_{\mathcal{F}}$ be the characteristic idempotent for $\mathcal{F}$. The rank of the matrix $(|P\backslash\omega_{\mathcal{F}}/Q|)_{P,Q\leq_{\mathcal{F}} S}$ is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.*

The bisets appearing in Theorems 1.3 and 1.4 play an important role in the theory of fusion systems. See [4] for more details.

We will give a proof of Theorem 1.1 (and hence obtain Theorem 1.2 as a corollary), which is slightly different from that of [6]. This new proof uses (at least explicitly) only the Burnside ring $B(G)$ of G, not the rational representation ring $R_{\mathbb{Q}}(G)$ as in [6]. Therefore it is better suited for adapting to the fusion system case (Theorems 1.3 and 1.4), which we do subsequently.

2 The group case

We prove Theorem 1.1. As remarked in Section 1, Theorem 1.2 then immediately follows as a corollary by [3].

Let G be a finite group. Let $B(G)$ be the Burnside ring of G, i.e., the Grothendieck ring of the isomorphism classes $[X]$ of finite G-sets X. As an additive group, $B(G)$ is a free abelian group with the canonical basis $\{[G/P] \mid P \leq_G G\}$. Let $\mathbb{Q}B(G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)$ and regard $B(G)$ as a subring of $\mathbb{Q}B(G)$. In particular the canonical basis for $B(G)$ is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$.

It is a well-known fact that for each $P \leq G$ the fixed-point map

$$\chi_P\colon B(G) \to \mathbb{Z}, \quad [X] \mapsto |X^P|,$$

is a ring homomorphism which depends only on the G-conjugacy class of P, and the product of these homomorphisms (tensored with $\mathbb{Q}$),

$$\chi = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{P \leq_G G} \chi_P\colon \mathbb{Q}B(G) \to \prod_{P \leq_G G} \mathbb{Q},$$

is a $\mathbb{Q}$-algebra isomorphism. For each subgroup $P \leq G$, let e_P^G denote the element of $\mathbb{Q}B(G)$ such that

$$\chi_Q(e_P^G) = \begin{cases} 1, & P =_G Q, \\ 0, & \text{otherwise.} \end{cases}$$

Then again the element e_P^G depends only on the G-conjugacy class of P and $\{e_P^G \mid P \leq_G G\}$ is a set of pairwise orthogonal primitive idempotents of $\mathbb{Q}B(G)$ whose sum is equal to 1; in particular it is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$. Furthermore, for $H \leq G$, let $B(G)_H$ be the subgroup of $B(G)$ generated by the elements $[G/P]$ with $P \leq_G H$. Then $\mathbb{Q}B(G)_H = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)_H$ is a subalgebra of $\mathbb{Q}B(G)$ with $\mathbb{Q}$-basis $\{[G/P] \mid P \leq_G H\}$. Note that the elements e_P^G with $P \leq_G H$ belong to $\mathbb{Q}B(G)_H$ and hence $\{e_P^G \mid P \leq_G H\}$ is another basis for $\mathbb{Q}B(G)_H$.

For each $P \leq G$ consider the $\mathbb{Q}$-linear map

$$\rho_P\colon \mathbb{Q}B(G) \to \mathbb{Q}, \quad [X] \mapsto |P\backslash X|,$$

which counts the P-orbits. By Burnside's orbit counting lemma, we have

$$\rho_P(x) = \frac{1}{|P|} \sum_{u \in P} \chi_{\langle u \rangle}(x), \quad x \in \mathbb{Q}B(G).$$

Thus

$$\rho_P(e_Q^G) \neq 0 \iff Q \text{ is cyclic and } Q \leq_G P. \tag{2.1}$$

Now the given matrix in Theorem 1.1 is equal to $(\rho_P(G/Q))_{P,Q\le_G H}$. By change of basis, this matrix has the same rank as $(\rho_P(e_Q^G))_{P,Q\le_G H}$. List the subgroups of H (up to G-conjugacy) in two families, the first consisting of cyclic subgroups and the second of noncyclic subgroups, and with nondecreasing order in each family. Then by (2.1) the above matrix has the form $\left(\begin{smallmatrix} A & 0 \\ B & 0 \end{smallmatrix}\right)$ where A is a lower triangular matrix with nonzero diagonal entries. Thus Theorem 1.1 follows.

3 The fusion system case

We first prove Theorem 1.3. In fact, we prove a slightly generalized version of it.

Proposition 3.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let Ω be a finite (S,S)-biset which is $\mathcal{F}$-stable and $\mathcal{F}$-generated and which contains the obvious (S,S)-biset S. Then the rank of the matrix $(|P\backslash\Omega/Q|)_{P,Q\le_{\mathcal{F}} S}$ is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.*

We first explain the terminology. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. An S-set X is *$\mathcal{F}$-stable* if, for all $P \le S$ and all morphisms $\varphi\colon P \to S$ in $\mathcal{F}$, the restrictions of the S-action on X to P via the inclusion $P \hookrightarrow S$ and via $\varphi\colon P \to S$ give isomorphic P-sets. We say that an (S,S)-biset is *$\mathcal{F}$-stable* if it is $\mathcal{F}\times\mathcal{F}$-stable viewed as a left $S\times S$-set by inverting the right action of S. An (S,S)-biset is *$\mathcal{F}$-generated* if, viewed as a left $S\times S$-set, all its isotropy subgroups are of the form $\Delta(\varphi,P)=\{(\varphi(u),u)\mid u\in P\}$ with $P\le S$, $\varphi\colon P\to S$ in $\mathcal{F}$. A finite (S,S)-biset Ω is called a *characteristic biset* for $\mathcal{F}$ if it is $\mathcal{F}$-stable and $\mathcal{F}$-generated and such that $|\Omega|/|S|$ is not divisible by p. It is easy to see that every characteristic biset Ω contains the (S,S)-biset S.

Define

$$B(\mathcal{F}) = \{x\in B(S)\mid \chi_P(x)=\chi_{P'}(x) \text{ for all } P,P'\le S \text{ with } P=_{\mathcal{F}} P'\}.$$

Clearly $B(\mathcal{F})$ is a subring of $B(S)$, which is called the Burnside ring of the fusion system $\mathcal{F}$. For a finite S-set X, we have $[X]\in B(\mathcal{F})$ if and only if X is $\mathcal{F}$-stable. The elements

$$e_P^{\mathcal{F}} := \sum_{P'=_{\mathcal{F}}P} e_{P'}^S,$$

where $P\le S$ and the sum is over the S-conjugacy classes of subgroups P' of S which are $\mathcal{F}$-conjugate to P, belong to $\mathbb{Q}B(\mathcal{F})$. The set $\{e_P^{\mathcal{F}}\mid P\le_{\mathcal{F}} S\}$ is a set of pairwise orthogonal primitive idempotents of $\mathbb{Q}B(\mathcal{F})$ whose sum is equal to 1; in particular it is a $\mathbb{Q}$-basis for $\mathbb{Q}B(\mathcal{F})$. By (2.1), we have

$$\rho_P(e_Q^{\mathcal{F}})\ne 0 \iff Q \text{ is cyclic and } Q\le_{\mathcal{F}} P. \tag{3.1}$$

Let Ω be the (S, S)-biset given in the above proposition. By the $\mathcal{F}$-stability of Ω, the left S-set Ω/P of the right P-orbits of Ω is also $\mathcal{F}$-stable for $P \leq S$. Moreover

$$\chi_Q([\Omega/P]) \neq 0 \implies Q \leq_{\mathcal{F}} P; \quad \chi_P([\Omega/P]) \geq |N_S(P)/P|.$$

The former implication follows from the fact that Ω is $\mathcal{F}$-generated and the latter inequality comes from the fact that Ω contains S. Hence $\{[\Omega/P] \mid P \leq_{\mathcal{F}} S\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}B(\mathcal{F})$. Thus the matrix

$$(|P\backslash\Omega/Q|)_{P,Q\leq_{\mathcal{F}}S} = (\rho_P([\Omega/Q]))_{P,Q\leq_{\mathcal{F}}S}$$

has the same rank as $(\rho_P(e_Q^{\mathcal{F}}))_{P,Q\leq_{\mathcal{F}}S}$, which is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S by (3.1).

Remark. Note that the finite group G in Theorem 1.2, viewed as an (S, S)-biset, satisfies the hypotheses for Ω in Proposition 3.1. Thus Theorem 1.2 can also be obtained from Proposition 3.1.

Now we address Theorem 1.4. In Proposition 3.1, the condition that the finite (S, S)-biset Ω contains the (S, S)-biset S is equivalent to that $\chi_P(\Omega/P) \neq 0$ for all $P \leq S$, given the other conditions on Ω. Proposition 3.1 then applies to all virtual (S, S)-bisets ω with coefficients in $\mathbb{Q}$ which are $\mathcal{F}$-stable, $\mathcal{F}$-generated and such that $\chi_P(\omega/P) \neq 0$ for all $P \leq S$, where ω/P denotes the linearized right P-orbits of ω. The proof is identical to the one given above. In particular, Reeh [5, Proposition 4.5, Corollary 5.8] shows that if ω is the *characteristic idempotent* of $\mathcal{F}$, i.e., the unique virtual (S, S)-biset with coefficients in $\mathbb{Z}_{(p)}$ which is $\mathcal{F}$-stable, $\mathcal{F}$-generated and which is an idempotent in the double Burnside ring $\mathbb{Z}_{(p)}B(S, S)$ of S (i.e., the Burnside ring of finite (S, S)-bisets), then the elements $\omega/P = \omega \circ_S [S/P] = \beta_P$ with $P \leq_{\mathcal{F}} S$ form a basis of $\mathbb{Z}_{(p)}B(\mathcal{F})$ such that $\chi_P(\omega/P) \neq 0$. This proves Theorem 1.4.

Acknowledgments. We thank Professor Markus Linckelmann for pointing us to the paper of Thévenaz and Professor Thévenaz for helpful discussions.

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Received August 27, 2013; revised November 26, 2013.

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